final

DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A2 Mathematical Methods II

HILARY TERM 2016

THURSDAY, 21 APRIL 2016, 9.30am to 11.30am

This exam paper contains three sections.
Candidates should submit answers to a maximum of **four** *questions for credit that include an answer to at least* **one** *question in each section.*

Please start the answer to each question in a new answer booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Nonlinear Systems

1. (a) [4 marks] Define what is meant by an equilibrium solution for the system of differential equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^3$. Describe how to find its linear stability.

(b) [14 marks] The self-exciting Faraday disk dynamo is described by the nonlinear equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (\bar{\alpha} - 1)x - xy - \beta z,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \kappa(\bar{\alpha}x^2 - y),$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - \lambda z,$$

where α, β, κ and λ are positive parameters with $\bar{\alpha} = \alpha/\kappa$ and $\bar{\beta} = \beta/\lambda$.

(i) Determine all the equilibrium solutions. Show that pitchfork and Hopf bifurcations are possible for the origin, and find equations for $\bar{\alpha}$ in terms of $\bar{\beta}$, κ and λ for these to occur. Determine the frequency of the bifurcating limit cycle. Show that it is also possible for two eigenvalues to pass through zero simultaneously. What happens to the frequency of the limit cycle in this case? Sketch the curves of pitchfork and Hopf bifurcations in the $(\bar{\beta}, \bar{\alpha})$-plane.

(ii) By constructing a suitable Lyapunov function, show that the origin is globally asymptotically stable provided $\bar{\alpha} < 1$.

(c) [7 marks] For the non-zero equilibrium solutions, show that curves of pitchfork bifurcation coincide with those for the zero equilibrium. Show, also, that two eigenvalues can also pass through zero for the same conditions as for the zero equilibrium. Show that Hopf bifurcations occur along the curve

$$\bar{\alpha_H} = \frac{\lambda(2\bar{\beta} - \kappa - \lambda)}{2(\kappa - \bar{\beta})} + \frac{3}{2}\bar{\beta} + 1,$$

provided $\bar{\beta} \neq \kappa$ and $\bar{\alpha} + \frac{\lambda}{2} > 1 + \frac{3\bar{\beta}}{2}$.

2. (a) [5 marks] Show that the system of nonlinear equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x - y + x^3,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + \mu y - xy$$

has an equilibrium state at the origin, which undergoes a Hopf bifurcation at $\mu = 0$.

(b) [15 marks] By perturbing about the equilibrium solution at the origin, use the Poincaré-Lindstedt method to find periodic solutions when $\mu = -\epsilon^2$, with $0 < \epsilon \ll 1$. [*You might find it useful to define $z = x + iy$.*]

(c) [5 marks] Show that the amplitude of the periodic solution is approximately $2\sqrt{-2\mu/3}$ and the frequency is approximately $1 + \mu/9$.

Section B: Further Mathematical Methods

3. (a) [15 marks] Solve the equation

$$y(x) = 2 - 12x^2 + \lambda \int_0^1 (1 - 3xt) y(t) \, \mathrm{d}t.$$

Be careful to consider all values of λ and give all solutions when multiple solutions exist.

(b) [10 marks] For which real values of the constant λ does the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda y - 2xy = f(x) \qquad \text{on } 0 \leqslant x \leqslant 1 \qquad \text{with } y(0) = y(1)$$

have a unique solution? When λ is such that the solution is not unique, under what condition on $f(x)$ does a solution exist?

4. (a) [12 marks] Suppose the function $u(t)$ is the optimal control which minimises the cost functional

$$C[x, u] = \int_0^T h(t, x, u) \, \mathrm{d}t$$

over all controls $u(t) \in C^1[0, T]$ satisfying the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, u), \qquad x(0) = a, \quad x(T) = b,$$

where $\partial f / \partial u \neq 0$. Show that u satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial u} \bigg/ \frac{\partial f}{\partial u}\right) = \frac{\partial h}{\partial x} - \frac{\partial f}{\partial x}\left(\frac{\partial h}{\partial u} \bigg/ \frac{\partial f}{\partial u}\right).$$

(b) [6 marks] A process obeys the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + u, \qquad x(0) = 1, \quad x(1) = 0,$$

and it is desired to minimise the integral

$$C[x, u] = \int_0^1 (u(t)^2 + 3x(t)^2) \, \mathrm{d}t.$$

Show that

$$u = \frac{-3\mathrm{e}^{4-2t} - \mathrm{e}^{2t}}{\mathrm{e}^4 - 1},$$

and find the corresponding behaviour of x.

(c) [7 marks] Now suppose that the requirement $x(1) = 0$ is removed, leaving only the condition $x(0) = 1$. Show that the natural boundary condition at $t = 1$ is

$$\dot{x}(1) = x(1).$$

Show that the optimal solution for x is now

$$x = \frac{3\mathrm{e}^{2t} + \mathrm{e}^{4-2t}}{3 + \mathrm{e}^4},$$

and find the corresponding control u.

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Section C: Further PDEs

5. (a) (i) [5 marks] Define the Hankel transform $\mathcal{H}[f(r);k]$ of a function $f(r)$. Show that

$$\mathcal{H}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}; k\right] = -k^2\mathcal{H}[f(r);k].$$

(ii) [8 marks] Let $u(x,y,t)$ satisfy the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u, \qquad -\infty < x, y < \infty, \quad t \geqslant 0,$$

with $u \to 0$ as $x^2+y^2 \to \infty$ and initial condition $u(x,y,0) = f(r)$ where $r = \sqrt{x^2+y^2}$, and

$$f(r) = \begin{cases} 1 & \text{for } r \leqslant a, \\ 0 & \text{for } r > a. \end{cases}$$

Show that

$$u(r,t) = a\int_0^\infty J_1(ka)J_0(kr)\mathrm{e}^{-k^2 t}\,\mathrm{d}k.$$

[*Note that Bessel functions satisfy the differential equation*

$$J_n''(x) + \frac{1}{x}J_n'(x) + \left(1 - \frac{n^2}{x^2}\right)J_n(x) = 0,$$

and the recurrence relations

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \qquad \frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x).]$$

(b) The Mellin transform is given by

$$\mathcal{M}[f(x);s] = F(s) = \int_0^\infty x^{s-1}f(x)\,\mathrm{d}x,$$

which exists in some strip $s_1 < s < s_2$. The inversion is given by

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s}F(s)\,\mathrm{d}s,$$

where $s_1 < c < s_2$.

(i) [3 marks] Show that for $a > 0$, $\mathcal{M}[f(ax);s] = a^{-s}\mathcal{M}[f(x);s]$.

(ii) [9 marks] For $x > 0$, let

$$G(x) = \sum_{k=1}^\infty k\mathrm{e}^{-k^2 x}.$$

Show that

$$G(x) = \frac{1}{2x} - \frac{1}{12} + o(1)$$

as $x \to 0+$.

[*You may use without proof the fact that the gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

has poles at $x = -m$ ($m = 0, 1, 2, \ldots$) with residue $(-1)^m/m!$. The Riemann zeta function, defined by

$$\zeta(x) = \sum_{n=1}^\infty n^{-x}$$

for Re$(x) > 1$, may be analytically continued to a meromorphic function which has a single pole at $x = 1$ with residue 1. Note also that $\zeta(-1) = -1/12$.]

6. Let the operator L be given by

$$Lu = -x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

on the interval $0 \leqslant x < \infty$ with the condition $u(0) = 0$.

(a) [10 marks] Show that the Green's function for $Lu - \mu u$ is

$$G(x,\xi;\mu) = \begin{cases} \dfrac{\mathrm{i}}{2\xi\sqrt{\mu}}\left(\dfrac{\xi}{x}\right)^{\mathrm{i}\sqrt{\mu}} & 0 \leqslant x < \xi < \infty, \\ \dfrac{\mathrm{i}}{2\xi\sqrt{\mu}}\left(\dfrac{x}{\xi}\right)^{\mathrm{i}\sqrt{\mu}} & 0 \leqslant \xi < x < \infty, \end{cases}$$

where you should define the branch of the square root.

(b) [10 marks] Use this to find the corresponding spectral representation of the delta function. Show that the cases $x > \xi$ and $x < \xi$ both give the same result.

(c) [5 marks] Deduce from (b) the Mellin transform pair

$$F(s) = \int_0^\infty \xi^{s-1} f(\xi)\,\mathrm{d}\xi, \qquad f(x) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} x^{-s} F(s)\,\mathrm{d}s.$$